

Lagrangian shadows of ample algebraic divisors

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Abstract

In the framework of Special Bohr - Sommerfeld geometry it was established that an ample divisor in compact algebraic variety can define almost canonically certain real submanifold which is lagrangian with respect to the corresponding Kahler form. It is natural to call it “lagrangian shadow”; below we emphasize this correspondence and present some simple examples, old and new. In particular we show that for irreducible divisors from the linear system $|\frac{1}{2}K_{F^3}|$ on the full flag variety F^3 their lagrangian shadows are Gelfand - Zeytlin type lagrangian 3 - spheres.

In preprint [1] one proposes a new programme which was called Special Bohr - Sommerfeld geometry, applicable in the broadest context to compact simply connected symplectic manifolds with integer symplectic forms. The most interesting particular case is given by the consideration of a compact simply connected algebraic variety X with an ample line bundle L . Then there exists a Kahler form ω on X with the integer cohomology class, equals to $c_1(L) \in H^2(X, \mathbb{Z})$ so after fixing a hermitian structure on L one can apply Special Bohr - Sommerfeld geometry and get the moduli space of special Bohr - Sommerfeld lagrangian cycles, see [1]. Recall that this set is formed by Bohr - Sommerfeld lagrangian cycles which are special with respect to holomorphic sections of L .

Now suppose that we have an ample divisor $D \subset X$ in a compact simply connected algebraic variety X of complex dimension n . By the very definition, see [2], the complete linear system $|kD|$ for certain $k \in \mathbb{N}$ gives an embedding $\phi : X \rightarrow \mathbb{CP}^N$ to the projective space. Fixing a standard Fubini - Study metric on \mathbb{CP}^N one gets a Kahler metric of the Hodge type on X such that the Kahler form ω_k has the cohomology class Poincare dual to $k[D] \in H_{2n-2}(X, \mathbb{Z})$; and at the same time it gives a hermitian structure $|\cdot|_k$ on the line bundle $L_{kD} = \phi^*\mathcal{O}(1) \rightarrow X$ which corresponds to the standard hermitian structure on $\mathcal{O}(1) \rightarrow \mathbb{CP}^N$ coming after the fixing of the Kahler structure on the last projective space.

Consider the holomorphic section $h_D \in H^0(X, L_{kD})$ with the multiple zero locus $(h_D)_0 = kD$; this section is defined up to scaling, but the following real function

$$\psi_D^k = -\ln|h_D|_k$$

is correctly defined on the complement $X \setminus D$. In the last expression for the function we indicate the level k since it is possible to consider the embeddings

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of different degrees and the result of our construction evidently depends on the degree.

Since the Kahler metric with the Kahler form ω_k has been fixed on X one can consider the gradient flow generated by the function ψ_D^k on the complement $X \setminus D$. Since D is the region where ϕ_D^k goes to $+\infty$ the behavior of the gradient flow near D is clear, so D can be understood as “infinite maximum set” for the function ψ_D^k .

In [1] it was shown that a Bohr - Sommerfeld lagrangian submanifold $S \subset X \setminus D$ is special w.r.t. the holomorphic section h_D iff S is stable with respect to the gradient flow generated by ψ_D^k :

$$F_{\psi_D^k}^t(S) = S \quad \text{for any } t.$$

Therefore in this situation we are interested in other “finite” critical points of ψ_D^k and in the “final” trajectories of the gradient flow $F_{\psi_D^k}^t$. At a non critical point the trajectories can’t form a local submanifold of dimension greater than n : according to Milnor, see [3], every “finite” critical point of ψ_D^k must have the Morse index less or equal to n (since ψ_D^k is strictly convex on $X \setminus D$ and plurisubharmonic w.e.t. ω_k , see [1]). On the other hand, this local submanifold must be isotropical due to the same reason, therefore if ψ_D^k admits “finite” critical points of Morse index n then the “finite” trajectories give us some *lagrangian* patches, and in good cases one can form lagrangian submanifolds or lagrangian cycles from the set of these patches.

We would like to call such a submanifold or cycle **lagrangian shadow of level k** for a given ample algebraic divisor D and denote it as $Sh^{Lag}(D)$. Indeed, it is defined almost uniquely: we have fixed in the construction above only a standard Fubini - Study metric on the projective space, but the shadows in the real life as well are not unique — they depend on the Sun’s position in the sky (in particular sometimes shadows vanish...). The main interest in the presented construction follows from the fact that we start with pure algebraic situation (algebraic variety, ample divisor) and get something from other part of the realm of Geometry — lagrangian.

Today this correspondence has just a conjectural explanation, therefore we are speaking about it as phenomenological observations. The first two examples have been presented in [1]:

Example 0. Let X be \mathbb{CP}^1 with the standard Fubini -Study metric and the degree of D is 2. Then if D is an irreducible element of $|2h|$ then its lagrangian shadow $Sh^{Lag}(D) = S^1$ is a circle; if D is reducible, then the shadow doesn’t exist (see [1]).

Example 1. Let X be the complex quadric $\mathbb{CP}^1 \times \mathbb{CP}^1$ with the standard Kahler structure, and D has bi - degree (1,1) (so $L_D = \mathcal{O}(1,1)$). Then if D is irreducible then $Sh^{Lag}(D) = S^2$, and this lagrangian sphere is Hamiltonian isotopic to the antidiagonal embedding $S^2 = \{[x_0 : x_1] \times [\bar{x}_0 : \bar{x}_1]\}$; if D is reducible then the lagrangian shadow doesn’t exist (see [1]). In this example we have homological non triviality of the lagrangian shadow which present in this case the class $(h, -h)$ up to sign.

Example 1a. Generalizing the previous example, take $X = \mathbb{CP}^n \times \mathbb{CP}^n$ with the standard product Kahler structure. Let D be the zeroset of a holomorphic section of the bundle $\mathcal{O}(1,1)$ then again if D is irreducible then $Sh^{Lag}(D)$ is isomorphic to \mathbb{CP}^n , embedded to the product in the same way: $\{[z_0 : \dots :$

$z_n] \times [\bar{z}_0 : \dots : \bar{z}_n]\}$ and then deformed by a Hamiltonian isotopy. The arguments are the same as in **Example 1**.

Example 1b. Another generalization of **Example 1** is to consider the product $Q_k = \mathbb{CP}^1 \times \dots \times \mathbb{CP}^1$ with the bundle $\mathcal{O}(1, \dots, 1)$ where the number of summands is $k > 2$. The computations in this case are much more complicated than in Example 1, but several facts can be established in short.

Take Q_3 with the bundle $\mathcal{O}(1, 1, 1)$ and fix homogenous coordinates $[x_0 : x_1], [y_0 : y_1], [z_0 : z_1]$ on each \mathbb{CP}^1 . Take the section h_D of $\mathcal{O}(1, 1, 1)$ given by the polynomial $x_0 y_0 z_0 + x_1 y_1 z_1$. Then the function ψ_D^1 has the following finite critical points and sets: 1) two minimal points at $[1 : 0] \times [1 : 0] \times [1 : 0]$ and (symmetrically) $[0 : 1] \times [0 : 1] \times [0 : 1]$; 2) critical set $T_c = \{[1 : e^{i\eta_x}] \times [1 : e^{i\eta_y}] \times [1 : e^{i\eta_z}] | \eta_x + \eta_y + \eta_z = 0\}$ which is isomorphic to 2 - torus. Taking all gradients lines which start at the minimal points and run to T_c we get a singular submanifolds $S \subset Q_3$, by the very definition this S is the lagrangian shadow $Sh^{Lag}(D)$ of the irreducible divisor $D = \{x_0 y_0 z_0 + x_1 y_1 z_1 = 0\}$. Geometrically this S is isomorphic to 3 - dimensional sphere with two singular points appear after the shrinking of two unknotted smooth circles. Using the toric actions on the direct summands in Q_3 we can extend this observation to generic smooth divisors from the same complete linear system. On the other hand for the reducible divisors the lagrangian shadows vanish. It follows from the fact that if D is reducible then h_D is the tensor product of holomorphic sections of $\mathcal{O}(1, 1, 0)$ and $\mathcal{O}(0, 0, 1)$ up to transpositions of 1's, therefore the function ϕ_D^1 is the sum of two functions and it does admit no finite critical points of Morse index 3. But as it was explained in [1] for the existence of special Bohr - Sommerfeld cycle one needs critical points of this index.

The situation in the case of Q_4 is much reacher: in this case even for reducible divisors from the complete linear system $|h_1 + \dots + h_4|$ where h_i is the generator for i 'th summand, one has nontrivial lagrangian shadows. Indeed, let us take $h_D = h_{D_1} \otimes h_{D_2}$ where $D_1 \in |h_1 + h_2|, D_2 \in |h_3 + h_4|$ and both D_i are irreducible. Then for each D_i we get, according to **Example 1**, the lagrangian shadows lifted from the first and the second pairs of direct summands in Q_4 and totally for D we get $Sh^{Lag}(D)$ isomorphic to the direct product $S^2 \times S^2$ of 2 - spheres. The same is true, of course, for any other division of the set $\{h_i\}$ into pairs, but for the reducible divisors of other types (3+1) we again get vanishing lagrangian shadows. Irreducible divisors shadow essentially the same as in the case of Q_3 : the difference is in the structure of two singular points in $Sh^{Lag}(D)$.

Our last example is more geometrically interesting:

Example 2. Let X be the full flag variety F^3 for \mathbb{C}^3 with the standard Kahler structure coming from the realization of F^3 as the incidence cycle in the direct product $\mathbb{CP}^2 \times \mathbb{CP}^2$, and the line bundle $L = \mathcal{O}(1, 1)|_X = K_X^{-\frac{1}{2}}$. Then we claim that for irreducible sections of the bundle the lagrangian shadows of the corresponding divisors is Hamiltonian equivalent to lagrangian 3 - sphere which is known from the Gelfand - Zeytlin system consideration, see [4] and references therein. This is also true for certain reducible divisors from the same linear system. Our arguments are based on the **Example 1a** above and on some functorial property of Special Bohr - Sommerfeld geometry.

Proposition. *Let $Y \subset X$ is a smooth algebraic subvariety in X , and $D_Y \subset Y$ is the intersection $D \cap Y$ for a very ample divisor $D \subset X$. Suppose that this intersection is transversal. Then if the lagrangian shadow $Sh^{Lag}(D) \subset$*

X intersects Y cotransversally so if $\dim_{\mathbb{R}}(\text{Sh}^{\text{Lag}}(D) \cap Y) = \dim_{\mathbb{C}} Y$ then this intersection equals to $\text{Sh}^{\text{Lag}}(D_Y)$ in Y .

The proof is direct: take the corresponding line bundle $L_D \rightarrow X$, fix the Kahler form ω , then take the corresponding hermitian connection on L_D . Then restricting all the data on Y we get that D_Y is defined by $h_D|_Y$, and the covariantly constant section of $(L_D, a)|_{\text{Sh}^{\text{Lag}}(D)}$ restricts to a covariantly constant section of $(L_D, a)|_{\text{Sh}^{\text{Lag}}(D) \cap Y}$, so the proportionality coefficient function used to be real positive after any restrictions, which gives us the specialty condition for the intersection $\text{Sh}^{\text{Lag}} \cap Y$ with respect to the holomorphic section $h_D|_Y$, so it remains only one important condition for $\text{Sh}^{\text{Lag}} \cap Y$ — to be lagrangian in Y , what essentially means that the intersection is *cotransversal*.

Now let $X = \mathbb{CP}^2 \times \mathbb{CP}^2$, algebraic subvariety $Y = F^3$ is given by the equation $\sum x_i y_i = 0$ for fixed homogenous coordinates on both \mathbb{CP}^2 's, and the divisor D is given by the polynomial $x_0 y_0 + x_1 y_1 - x_2 y_2 = 0$. Then it is not hard to see that $\text{Sh}^{\text{Lag}}(D)$ is defined by the relation $\{[x_0 : x_1 : x_2] \times [\bar{x}_0 : \bar{x}_1 : -\bar{x}_2]\}$ in X , being a copy of the projective plane. But according to [4] the intersection $\text{Sh}^{\text{Lag}}(D) \cap Y$ is exactly the Gelfand - Zeytlin lagrangian sphere; and therefore it is the lagrangian shadow of $D_Y = D \cap Y$. Now the point is that D_Y is reducible in $Y = F^3$: it is formed by two del Pezzo surfaces of degree 8, see [2]. However we can vary the section $h_{D_Y} \rightarrow h_{D_{ir}}$ taking the polynomial $\alpha_0 x_0 y_0 + \alpha_1 x_1 y_1 - \alpha_2 x_2 y_2$, and if the expression

$$\frac{\alpha_0 x_0 y_0 + \alpha_1 x_1 y_1 - \alpha_2 x_2 y_2}{x_0 y_0 + x_1 y_1 - x_2 y_2} \Big|_{\text{Sh}^{\text{Lag}}(D_Y)}$$

is real positive everywhere then $\text{Sh}^{\text{Lag}}(D_{ir}) = \text{Sh}^{\text{Lag}}(D_Y)$, see [1]. This happens if every $\alpha_i \in \mathbb{R}_+$, and this gives us big family of irreducible divisors with the same lagrangian shadow.

The arguments from the toric geometry lead to the claim that the same is true for generic irreducible divisor from the complete linear system $|(h_1 + h_2)|_{F^3}| = |-\frac{1}{2}K_{F^3}|$.

We end these notes with the following final remark¹. It was conjectured in [1] that under the variation of ample divisors in the complete linear system $|D|$ the “lagrangian shadows” keep stable: the “number” of components in $\text{Sh}^{\text{Lag}}(D)$ is the same. This conjecture is very naive, so in general setup it is much more reasonable to formulate such a stability property on the homological level. Since even in the best case a “lagrangian shadow” comes without any preferred orientation, we are speaking about mod \mathbb{Z}_2 version: the collection of components of a lagrangian shadow gives a class from $H_n(X, \mathbb{Z}_2)$. It seems that this class must be the same for generic elements of $|D|$. It implies that in the case when D is very ample (so $k = 1$) this class depends on the Kahler form ω_1 only. Does this class depend on the cohomology class of ω_1 rather than on the form itself? From the Hard Lefschetz we know that the cohomology class $[\omega_1]$ gives isomorphisms $H^{n-k} \rightarrow H^{n+k}$ but doesn't touch the middle part H^n , so may be there is some hidden part of the Hard Lefschetz theorem which concerns the middle cohomology group, and the story with lagrangian shadows just reflects this one.

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References

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